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EXAMPLES OF SEMIPERFECT RINGS†

BY
LOUIS HALLE ROWEN
Department of Mathematics, Bar Ilan University, Ramat Gan, Israel

ABSTRACT
By means of generic methods, an example is given of a local (but not Noetherian) π-regular ring R, over which the ring of 2×2 matrices is *not* π-regular. Also a cyclic indecomposable (left) module over a right Artinian ring is exhibited, whose endomorphism ring is not local.

Suppose R is a ring with Jacobson radical J. We say R is *semilocal* if R/J is semisimple Artinian. A semilocal ring R is *semiperfect* if J is idempotent-lifting; one important special case is when J is nil. In particular J is nil when R is *left π-regular*, i.e. if R satisfies DCC on all chains of the form $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots$ (cf. [5] for a more detailed discussion). By [5, Theorem 8], if R is a semiperfect ring all of whose matrix rings are left π-regular then there is a version of Fitting's lemma for finitely presented indecomposable R-modules, leading to an analogue of the Azumaya–Krull–Schmidt decomposition theory for R-modules. Also these conditions are necessary for such an Azumaya–Krull–Schmidt–Fitting theory. Thus left π-regular rings have a role in basic module theory.

Since the condition of left π-regularity is hard to verify, one is led to look for a more manageable criterion. By [5, Proposition 13] a semiperfect ring R is left π-regular if J equals the lower nilradical, leading one to ask if R is necessarily left π-regular whenever J is nil. As we shall see below this is not necessarily the case. Section 2 provides an example of a semiperfect (non-Noetherian) ring R whose Jacobson radical J is nil, but R is *not* left π-regular; in the process we see that the 2×2 matrix ring over a left π-regular ring need not be left π-regular, thereby answering negatively an old question in the theory of left π-regular rings. In fact we give several such examples, one for which J is *not* locally

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nilpotent and one for which J *is* locally nilpotent; the former relies on the Golod–Shafarevich theorem.

In studying semiperfect rings one might ask whether the Fitting's Lemma route is too stringent; perhaps one could obtain a reasonable Azumaya–Krull–Schmidt theory for arbitrary f.g. (finitely generated) modules if one were willing to prove directly that f.g. indecomposables are LE, i.e. the endomorphism ring is a local ring. This question was pointed out to me by G. Abrams, who showed that the known f.g. indecomposables are indeed LE. In §3 we see an example of a cyclic indecomposable (left) module over a right Artinian ring which is *not* LE.

§1. A matrix condition for semiperfect rings

We start by looking for a local ring R_0 whose Jacobson radical J is nil, but for which $R = M_2(R_0)$ is *not* left π-regular. Note R_0 automatically is left π-regular since each element is invertible or nilpotent, and R_0 is semiperfect since all local rings are semiperfect. Hence R is semiperfect.

To construct the ring R_0 we consider more generally an arbitrary ring R with a nontrivial idempotent e, and let R_0 be eRe. We say an element r is *proper* if we can write $r = x + a + b + c$ (where $x = ere$ is invertible in eRe, $a \in eJ(1-e)$, $b \in (1-e)Re$, and $c \in (1-e)J(1-e)$). Let us examine explicitly the conditions on the proper element r which are necessitated by $r \in Rr^2$. (Of course the intuition is to view r as the matrix $\begin{pmatrix} x & a \\ b & c \end{pmatrix}$.)

Take y in eRe satisfying $xy = yx = e$. Clearly we must have $r = sr^2$ for some s in R; letting $u = (1-e)se$ and $v = (1-e)s(1-e)$ (i.e. intuitively the "bottom row" of s is (uv)) we match entries in the "bottom row" of r and of sr^2 to get

$$b = u(x^2 + ab) + v(bx + cb), \tag{1}$$

$$c = u(xa + ac) + v(ba + c^2). \tag{2}$$

Multiplying each side of (1) by ya yields

$$bya = u(xa + abya) + v(ba + cbya). \tag{3}$$

Subtracting (3) from (2) now yields

$$c - bya = ua(c - bya) + vc(c - bya),$$

so

$$(1 - ua - vc)(c - bya) = 0.$$

Now $ua + vc \in (1-e)J(1-e)$, the Jacobson radical of $(1-e)R(1-e)$, so $1 - ua - vc$ is invertible in $(1-e)R(1-e)$, yielding $c - bya = 0$, i.e. $c = bya$. We shall say r is *degenerate* if this condition $c = bya$ holds on the "entries" of r. Thus we have shown that $r \notin Rr^2$ if r is nondegenrate.

§2. Computing nondegeneracy

GENERAL FACT 2.1. If r is invertible and a is in J then $r + a = r(1 + r^{-1}a)$ is invertible.

Now let r be "proper", as defined in §1. Then r^2 also is proper; for example $er^2e = x^2 + ab$ is invertible in eRe since $ab \in eJe$. Thus $Rr > Rr^2 > Rr^3 > \cdots$ if each power of r is nondegenerate. (This is seen by noting that $Rr > Rr^2 > Rr^3 > \cdots$ iff $Rr > Rr^2 > Rr^4 > \cdots$.) By the same argument we need only verify r^n is nondegenerate for an infinite number of n.

View r^n as $\begin{pmatrix} x(n) & a(n) \\ b(n) & c(n) \end{pmatrix}$; note $x(1) = x$, $a(1) = a$, $b(1) = b$, and $c(1) = c$. Viewing these entries as polynomials in x, a, b, c we write

$$a(n) = \sum_{j(1)=0}^{n-1} x^{n-1-j(1)} ac^{j(i)} + \sum_{i=0}^{n-3} \sum_{j(1)=0}^{n-3-i} \sum_{j(2)=0}^{n-3-i-j(1)} x^i ac^{j(1)} bx^{n-i-j(1)-j(2)-3} ac^{j(2)}$$

$$+ \sum x^{i(1)} ac^{j(1)} bx^{i(2)} ac^{j(2)} bx^{n-i(1)-i(2)-j(1)-j(2)-j(3)-5} ac^{j(3)} + \cdots,$$

$$b(n) = \sum c^{j(1)} bx^{n-1-j(1)} + \sum c^{j(1)} bx^i ac^{j(2)} bx^{n-i-j(1)-j(2)-3}$$

$$+ \sum c^{j(1)} bx^{i(1)} ac^{j(2)} bx^{i(2)} ac^{j(3)} bx^{n-i(1)-i(2)-j(1)-j(2)-j(3)-5} + \cdots,$$

$$c(n) = c^n + \sum c^{j(1)} bx^{n-j(1)-j(2)-2} ac^{j(2)} + \sum c^{j(1)} bx^i ac^{j(2)} bx^{n-i-j(1)-j(2)-4} a$$

$$+ \sum c^{j(1)} bx^{i(1)} ac^{j(2)} bx^{i(2)} ac^{j(3)} bx^{n-i(1)-i(2)-j(1)-j(2)-j(3)-6} a + \cdots,$$

$$x(n) = x^n + \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-1} x^i ac^j bx^{n-i-j-2}$$

$$+ \sum x^{i(1)} ac^{j(1)} bx^{i(2)} ac^{j(2)} bx^{n-i(1)-i(2)-j(1)-j(2)-4} + \cdots$$

(where the second sum ranges over positive $i(1)$, $i(2)$, $j(1)$, $j(2)$ with $i(1) + i(2) + j(1) + j(2) \leqq n - 4$.)

Only a finite number of summands occur in each expression, since the length of each summand as a word is n, and the degree in a and b increases by 2 in each partial sum. Let us rewrite

$x(n) =$

$$\left(1 + \sum x^i ac^j bx^{-i-j-2} + \sum x^{i(1)} ac^{j(1)} bx^{i(2)} ac^{j(2)} bx^{-i(1)-i(2)-j(1)-j(2)-4} + \cdots\right) x^n$$

where here and in the sequel x^{-1} formally denotes y. We can write $x(n)$ as $(1 + d)x^n$ where d is the grand sum of all the terms in these summations. Note $d \in J$ and thus d is nilpotent.

Let $y(n)$ be the inverse of $x(n)$ in eRe, i.e. $y(n)x(n) = x(n)y(n) = e$. To compute $y(n)$ note that d is nilpotent so $(1 + d)^{-1} = 1 - d + d^2 - d^3 \pm \cdots$ and thus

$$y(n) = y^n(1 - d + d^2 + \cdots) = y^n\left(1 - \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} x^i ac^j bx^{-i-j-2} + \cdots\right).$$

To stress the appearance of y^n in these expressions we rewrite

$$y(n) = y^n(1 - d + d^2 + \cdots) = y^n\left(1 - \sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} x^i ac^j by^n x^{n-i-j-2} + \cdots\right).$$

If we opened up the parentheses we would see that each term of $y(n)$ starts with y^n. Furthermore the sign of any such term alternates according to the number of times y^n appears.

Then we see (in ascending powers of a) that $b(n)y(n)a(n)$ equals

$$\sum c^{j(1)} bx^{n-j(1)-j(2)-2} ac^{j(2)} + \sum c^{j(1)} by^n x^{n-j(1)-1}\left(\sum x^i ac^{j(2)} bx^{n-i-j(2)-j(3)-3} ac^{j(3)}\right)$$
$$+\left(\sum c^{j(1)} bx^i ac^{j(2)} bx^{n-i-j(1)-j(2)-3}\right) \sum y^n x^{n-j(3)-1} ac^{j(3)}$$
$$-\sum c^{j(1)} bx^{n-j(1)-1} y^n \left(\sum x^i ac^{j(2)} by^n x^{n-i-j(2)-2}\right) \sum x^{n-j(3)-1} ac^{j(3)} + \cdots$$

and we call this entire expression $(*)$.

We want to determine R_0 so that $(*)$ does *not* equal $c(n)$ for suitable arbitrarily large n. In particular if $c^k = 0$ we take $n > k$. Thus we shall consider the formal difference of $(*)$ and $c(n)$, which we denote as $(**)$; in other words $(**)$ equals $b(n)y(n)a(n) - c(n)$.

REMARK 2.2. A careful analysis of $(**)$: Most of the terms of $(**)$ formally cancel each other out. Indeed, let us write a typical nonzero term in $(**)$ as a word in the form

$$h = h_1 x^{i(1)} \cdots h_t x^{i(t)} h_{t+1}$$

where each $i(u)$ is an integer and x does not "appear" in $h_1, \ldots, h_{t+1}$, i.e., $h_1 = c^{j(1)}b$, $h_u = ac^{j(u)}b$ for each $1 < u \leqq t$, $h_{t+1} = ac^{j(t+1)}$, with each $j(u) \geqq 0$. We shall determine how the word h can appear as a term in $b(n)y(n)a(n)$ or in $c(n)$. Note that $h \neq c^n$ since $c^n = 0$; thus $t \geqq 1$.

Viewing h as a word, we see that for h to appear in (**), each initial subword of h must be of non-negative degree $\leqq n$; such a word h will be called *admissible*. Conversely any admissible word can be written as a term in (**), according to the following procedure:

Let $h' = h_1$, $n' = \deg(h')$, $h'' = h_2 x^{i(2)} \cdots h_{t-1} x^{i(t-1)} h_t$, and $n''' = \deg(h_{t+1})$,

$$(4) \qquad h = (h'x^{n-n'})(y^n x^{n'+i(1)} h'' x^{i(t)+n'''-n})(x^{n-n'''} h_{t+1}) \in b(n)y(n)a(n).$$

If some $i(u) \geqq 0$ then another term equal to h also appears in (**), with the opposite sign, as follows:

If $i(1) \geqq 0$ let $h' = h_1 x^{i(1)} h_2$, $n' = \deg(h')$, $h'' = h_3 x^{i(3)} \cdots h_t$, $n''' = \deg(h_{t+1})$ and note

$$(5) \qquad h = (h'x^{n-n'})(y^n x^{n'+i(2)} h'' x^{i(t)+n'''-n})(x^{n-n'''} h_{t+1}).$$

(5) appears in (**) with the opposite sign as in (4), since we have produced one extra appearance of y^n. On the other hand, if $i(u) > 0$ for some $u > 1$ and we have already written h in (**) then we can rewrite

$$h = h' x^{i(u)} h''$$

where $h' = h_1 x^{i(1)} \cdots x^{i(u-1)} h_u$ and $h'' = h_{u+1} x^{i(u+1)} \cdots h_{t+1}$, and let $n' = \deg(h')$ and $n'' = \deg(h'')$ (defining the degree formally, viewing h', h'' as words in a, b, c, and x). Then $n = n' + i(u) + n''$, and we can obtain the following new term in (*) having opposite sign (since there is one extra appearance of y^n):

$$h' x^{n-n'} y^n x^{n-n''} h''.$$

Applying these arguments to each $i(u)$ in turn we see that each term h occurs 2^m times, half of the time " + " and half of the time " − ", where m is the number of u for which $i(u) \geqq 0$. Consequently the terms in (**) corresponding to h cancel each other out unless $i(u)$ is negative for each $1 \leqq u \leqq t$; we shall call such admissible terms "good".

We have shown that $b(n)y(n)a(n) - c(n)$ equals the sum of the good terms (since the others cancel). Let us reexamine such a nonzero good term. We can rewrite

$$h = h_1 x^{-i(1)} \cdots h_t x^{-i(t)} h_{t+1}$$

where now each $i(u)$ is positive. On the other hand, every subword must have non-negative degree $< n$, leading to the following conditions, letting $m_j = \deg h_i$:

$$m_1 \leqq n, \quad i(1) \leqq m_1, \quad m_1 + m_2 - i(1) \leqq n, \quad i(1) + i(2) \leqq m_1 + m_2, \quad \cdots \ .$$

Let us express these relations a bit differently. Write h as

$$by^{i(1)}ac^{i'(1)}by^{i(2)}ac^{i'(2)} \cdots by^{i(t)}ac^{i'(t)}.$$

Let $z_j = by^ja$ and $s_j = 2 - j = \deg(z_j)$; note $0 \leqq j \leqq n$ so $2 \geqq s_j \geqq 2 - n$ for each j. We see h is good iff $n \geqq s_{i(1)} + i'(1) + \cdots + s_{i(u)} \geqq 1$ for all u.

We want to prove that suitable examples have good terms (for arbitrarily large n) whose sum is nonzero. Actually we shall see that c is unnecesary, i.e. we may assume $c = 0 = i'(u)$ for all u.

EXAMPLE 2.3. A local ring R_0 whose Jacobson radical is nil, but not locally nilpotent, but such that $R = M_2(R_0)$ contains an element r such that r^n is nondegenerate for infinitely many n. (Thus R is not left π-regular.) We shall define r to be the matrix $\begin{pmatrix} x & a \\ b & 0 \end{pmatrix}$ for which the previous computations are generic. To this end let $F(X)$ denote the field of rational expressions in X, i.e. the field of fractions of the polynomial ring, and expand $\{X^i : i \in \mathbf{Z}\}$ to a base $\mathcal{B}$ of $F(X)$ over F. Let T be the free product of $F(X)$ with the free algebra $F\{A, B\}$ in noncommuting indeterminates A, B. We view T as "generalized" polynomials in A and B, where the coefficients (in $F(x)$) are interspersed throughout the monomials.

Let I be the ideal of T generated by A^2, B^2, Aw, and wB for all w in $\mathcal{B} - \{1\}$. Thus $\bar{T} = T/I$ is a ring in which we have $0 = \bar{A}^2 = \bar{B}^2 = \bar{A}w = w\bar{B}$. Furthermore I is a graded ideal of T, with respect to degree in A and B. Thus $\bar{T} = T/I$ is graded by degree in $\bar{A}$ and $\bar{B}$.

Let $Z_j = BX^{-j}A$ for $1 \leqq j \leqq 2$. Then clearly their images $\bar{Z}_j$ generate a free subalgebra $\bar{T}'$ of $\bar{T}$. By the Golod–Shafarevich theorem (cf. [8, Lemma 6.2.7 and Proof of Theorem 6.2.9]), there is an ideal P' of $\bar{T}'$, graded in $\bar{Z}_1$ and $\bar{Z}_2$, such that $\bar{T}'/P'$ is infinite dimensional over F, i.e. having nonzero terms of arbitrarily long length. But these terms all have total degree $\geqq 0$ in $\bar{A}, \bar{B}, \bar{X}$.

Let $V = \{BwA : w \in \mathcal{B} - \{X^{-1}, X^{-2}\}\}$, and let P be the ideal of T generated by I, V, and homogeneous representatives (in the Z_i) of the given homogeneous generators of P'. We claim that prove $R_0 = T/P$ is the desired example.

Letting a, b, x, z_1, z_2 denote the respective images of A, B, X, Z_1, Z_2, we shall prove first $\bar{T}' \cap \bar{P} = P'$. Indeed, since $\bar{I} = 0$, any element in $\bar{T}' \cap \bar{P}$ has the form

$$d = \sum f_i p_i g_i + h$$

where $f_i, g_i \in \bar{T}$, $p_i \in P'$, and $h \in \bar{T}\bar{V}\bar{T}$. Consider a nonzero "monomial" q of $f_i p g_i$ (in $\bar{A}$ and $\bar{B}$). In order for q to contribute nontrivially to a value in $\bar{T}$, it must start in $\bar{B}$. But p starts with $\bar{B}$, so $f_i p = 0$ unless f_i ends in $\bar{A}$; hence we may assume f_i starts with $\bar{B}$ and ends with $\bar{A}$, and so is a word in $\bar{Z}_1$, $\bar{Z}_2$, and $\bar{V}$. But $\bar{V} = 0$, so the f_i are words in $\bar{Z}_1$, $\bar{Z}_2$, and likewise the g_1 are words in $\bar{Z}_1$, $\bar{Z}_2$. But then $\Sigma f_i p_i g_i \in \bar{T}'P'\bar{T}' = P'$, so $h \in \bar{T}' \cap \bar{T}\bar{V}\bar{T} = 0$, as seen by the fact that the $\{BwA : w \in \mathscr{B}\}$ generate a free algebra. We conclude $d = \Sigma f_i p_i g_i \in P'$, as desired.

We have proved that R_0 canonically contains $\bar{T}'/P'$, and thus has nonzero terms of arbitrarily long length in z_1 and z_2, which are "good" (as defined earlier) since the total degree of these z_1 are nonnegative.

On the other hand, let J be the ideal of R_0 generated by a and b. We claim J is nil. Indeed any element of J can be written in the form

$$f(z_1, z_2) + w_1 a + b w_2 + ab$$

where w_1, w_2 are arbitrary elements in $F(x\cdot)$. Taking to the mth power yields

$$f(z_1, z_2)^m + w_1 a f(z_1, z_2)^{m-1} + b w_2 f(z_1, z_2)^{m-1} + (ab)^m$$
$$+ (ab)^{m-1} w_1 a + b w_2 (ab)^{m-1}$$

(since every other placement of the $w_1 a$ or $b w_2$ yields $a w_1 a$ or $b w_2 b$ which is 0); taking m large enough such that $f(z_1, z_2)^{m-1} = 0 = (ab)^{m-1}$ yields the claim.

Finally note R_0/J is isomorphic to the field $F(X)$, proving R_0 is local, and we have verified all the desired properties of R_0, establishing the desired example. Q.E.D.

EXAMPLE 2.4. With a few modifications in the previous example we can construct R_0 such that the (Jacobson) radical of $R = M_2(R_0)$ is nil; thus R will be a semiperfect ring whose radical is nil, but not locally nilpotent, and R is not left π-regular. (This is the question which motivated the paper.) Our modifications stem from the observation that it is enough to show that every set of four elements in J is nilpotent (for then $M_2(J)$ is nil), and Golod in fact found a nil, non-nilpotent algebra N generated by 5 elements, such that every subset of four elements is nilpotent, cf. [6, Theorem 6.2.9]. We repeat the notation of

Example 2.3, now letting T be the free product of $F(X)$ with the free algebra $F\{A, B, C\}$ in noncommuting indeterminates A, B, C.

Let I be the ideal of T generated by A^2, B^2, AB, BC, CA, C^4, AXB, Aw, wB, Cw, and wC for all w in $\mathcal{B} - \{1\}$. Thus $\bar{T} = T/I$ is a ring in which

$$0 = \bar{A}^2 = \bar{B}^2 = \bar{A}\bar{B} = \bar{B}\bar{C} = \bar{C}\bar{A} = \bar{A}\bar{X}\bar{B} = \bar{C}^4 = \bar{A}w = w\bar{B} = \bar{C}w = w\bar{C}.$$

Let $\bar{Z}_j = \bar{B}\bar{X}^{-j}\bar{A}\bar{C}^3$ for $1 \leqq j \leqq 5$. Then clearly these $\bar{Z}_j$ generate a free subalgebra $\bar{T}'$ of $\bar{T}$. As stated above, there is an ideal P' of $\bar{T}'$, graded in the $\bar{Z}_j$, such that $\bar{T}'/P'$ is infinite dimensional over F, i.e. having nonzero terms of arbitrarily long length, but such that every subset of four elements is nilpotent, so extending this as before to an ideal of $\bar{T}$ we see $R_0 = \bar{T}/P$ is the desired example.

A different flavor can be obtained by using an infinite number of Z_j; then we can also disregard the Golod–Shafarevich example.

EXAMPLE 2.5. A local ring R_0 whose Jacobson radical is locally nilpotent, but such that $M_2(R_0)$ contains an element r for which r^n is nondegenerate for infinitely many n. Thus $R = M_n(R_0)$ has Jac(R) locally nilpotent, but is not left π-regular.

We proceed as in the first four paragraphs of Example 2.3, but instead of working with only a finite number of Z_j, we shall use all $\{Z_j : j \in \mathbf{N}\}$ and take $V = \{BwA : w \in \mathcal{B} - \{X^{-j} : j \in \mathbf{N}\}\}$, and let P be the ideal of T generated by V and $\bigcup \{Z_1, \dots, Z_k\}^{(k+3)!}$. Thus any word of length $(k+3)!$ in $Z_1, \dots, Z_k$ is contained in P. $R_0 = T/P$ is certainly local as before, with J locally nilpotent, but has the "good" term

$$(((z_1^3 z_2)^4 z_3)^5 z_4)^6 \cdots$$

for arbitrarily large n, proving no power of r is degenerate.

The reason Example 2.5 "works" is that the index of nilpotence of $\{z_1, z_2, \dots, z_k\}$ rises so quickly (namely, on the order of $k!$). It would be very interesting to find some criterion on J which would assure that a semiperfect ring is left π-regular, in analogy to the fact that T-nilpotence implies that it is perfect.

§3. Finitely generated modules over perfect rings

The object of this section is to modify an example given in [6], to produce an indecomposable, cyclic module over a right Artinian PI-ring, which is not LE, i.e. its ring of endomorphisms is not local. Some observations also are made in order to put this example in its proper perspective; in particular we see "why" the prior examples in the literature are LE.

Recall a semilocal ring R is *left perfect* if its Jacobson radical J is T-nilpotent (in the sense that for every $a_1, a_2, \dots$ in J there is a suitable n with $a_1 \cdots a_n = 0$, cf. [3] or [6, Theorem 2.7.33]). Thus right Artinian implies perfect. Other important properties of perfect rings: Every R-module has a *projective cover*, i.e. has the form P/K where P is projective and K is a "small" submodule of P, written $K \ll P$ (i.e. if $K + N = P$ then $N = P$), cf. [6, Exercise 2.8.28]; for any natural number t, R satisfies the descending chain condition on submodules spanned by $\leqq t$ elements, cf. [2, Theorem 2]. Incidentally [2] is an excellent source for results concerning perfect rings; we also use [6] for a general reference.

First, let us see why all the previously recorded examples of modules over left perfect rings are LE. Throughout, assume M is a f.g. indecomposable module over a left perfect ring R.

LEMMA 3.1. *$f: M \to M$ is one-to-one iff f is an isomorphism.*

PROOF. Suppose f is one-to-one. If M is generated by t elements then so is f^iM for each i, implying $f^iM = f^{i+1}M$ for some i, by [2, Theorem 2]. Then for any y in M we have $f^iy = f^{i+1}x$ for suitable x in M, implying $y - fx \in \ker f^i = 0$, so $y = fx$, implying f is onto and thus an isomorphism. Q.E.D.

LEMMA 3.2. *The following condition is necessary and sufficient for M to be an LE-module: If $f + g = 1$ for $f, g: M \to M$ then* $\ker f = 0$ *or* $\ker g = 0$.

PROOF. Apply Lemma 3.1 to the usual criterion for a ring to be local ($a + b = 1 \Rightarrow a$ or b is invertible).

REMARK 3.3. If $f + g = 1$ then $(\ker f) \cap (\ker g) = 0$. We conclude any uniform f.g. R-module is LE, for then either $\ker f = 0$ or $\ker g = 0$.

REMARK 3.4. We can improve a bit on Remark 3.3. If $\ker f^i = \ker f^{i+1}$ for some i and $\ker f \neq 0$ then f is nilpotent (and thus $1 - f$ is invertible, so in particular $\ker(1 - f) = 0$).

PROOF. As in the proof of Lemma 3.1, $f^jM = f^{j+1}M$ for some j, and taking $j > 1$ we see by the usual Fitting's lemma argument [6, Proposition 2.9.7] that $M = f^jM \oplus \ker f^j$, implying $f^jM = 0$ since M is indecomposable. Q.E.D.

Thus, if $f + g = 1$ with $\ker f$, $\ker g$ nonzero we see $\ker f < \ker f^2 < \ker f^3 <$

$\cdots$, so defining $\ker^\infty f = \bigcup_{i=1} \ker f^i$, we see $\ker^\infty f$ is not f.g.; likewise $\ker^\infty g$ is not f.g. On the other hand, $\ker^\infty f \cap \ker^\infty g = 0$ for if $f^i x = 0 = g^j x$ for $0 \neq x \in M$ then we can find $y \neq 0$ such that $fy = gy = 0$, contrary to Remark 3.3. (Indeed we choose i, j minimal such, i.e. $f^{i-1}x \neq 0$ and $g^{j-1}x \neq 0$. Note that $g = 1 - f$ commutes with f; replacing x by $f^{i-1}x$ we may assume $fx = 0$; now let $y = g^{j-1}x$.) Thus we have shown

REMARK 3.5. If M is *not* LE then M has two submodules, neither of which is finitely generated, whose intersection is 0.

Bearing these observations in mind, we can find non-LE-modules without much difficulty. In [6, Exercise 2.7.22] we translated the idea of [4, Example 2.2] to matrices, and here it is boiled down to its essentials.

EXAMPLE 3.6. An indecomposable, non-LE, left module over a "nice" right Artinian ring R. First we define R. Let D be a division algebra over a field F, containing an element x transcendental over F. Let V be a right D-module of dimension 2, having base $\{y, z\}$, i.e. $V = yD + zD$. We also view V as left F-module, via multiplication $\alpha(yd_1 + zd_2) = y(\alpha d_1) + z(\alpha d_2)$. Clearly V is an $F - D$ bimodule, so $R = \begin{pmatrix} F & V \\ 0 & D \end{pmatrix}$ is a ring. R is semiprimary (in fact right but not necesarily left Artinian), and satisfies the following additional properties:

(i) Suppose D satisfies a polynomial identity (PI). Then R is a PI-ring; in particular if D is commutative then R satisfies the identity $(X_1X_2 - X_2X_1)^2$.

(ii) Suppose D is affine over F (i.e. $D = F\{d_1, \ldots, d_t\}$ for a finite set of elements $d_1, \ldots, d_t$ of D). Then R is affine, generated by e_{11}, $e_{12}y$, $e_{12}z$, and $e_{22}d_1, \ldots, e_{22}d_t$, where the e_{ij} are the standard matrix units.

Before continuing the example we pause for a few observations.

REMARK 3.7. Conditions (i) and (ii) cannot hold simultaneously, since any affine PI division algebra is finite dimensional by the Artin–Tate lemma, as formulated in [6, Corollary 6.3.2]. This contradicts the hypothesis that D contains an element transcendental over F.

Aside. Concerning condition (ii), the theory of semiprimary, nonartinian affine algebras may be void! Indeed consider the property that every affine division F-algebra is finite dimensional over F. If this holds then any affine semiprimary F-algebra is PI and algebraic over F, and thus finite dimensional over F, so that it is left and right Artinian. For F uncountable, all affine division F-algebras are algebraic by a result of Amitsur (cf. [6, Theorem 2.5.22]), so that we are left with Kurosh's problem for division algebras.

Now we construct M. Let L be the F-subspace of V spanned by all terms of the form yxh and $z(1-x)h$, where $h \in F[x]$. Let $M = \binom{V/L}{D}$, viewed as an R-module by matrix multiplication. M is a cyclic R-module since it is generated by $\binom{0}{1}$. Furthermore $Lx \subseteq L$, so right multiplication by x yields a map f: $V/L \to V/L$ which induces a map $f: M \to M$. But $y \in \ker f$ and $z \in \ker(1-f)$, so M is not LE. (If M were LE then f or $1-f$ must be an isomorphism.)

It remains to show M is indecomposable. To see this we shall show that if $g: M \to M$ with $g^2 = g$ then $g = 0$ or $g = 1$. Indeed writing $g\binom{0}{1} = \binom{a}{b}$ for suitable a in V/L and b in D, we see

$$\begin{pmatrix} a \\ b \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = g\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

so $a = 0$. But

$$g^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = g\begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b^2 \end{pmatrix}$$

so $b = b^2$ in the division ring D, implying $b = 0$ or $b = 1$. Hence $g = 0$ or $g = 1_M$, as desired.

Note that these ideas do not touch on the possibility of a Krull–Schmidt theory for indecomposables which are not LE.

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